ON HYPERSONIC FLOWS IN NOZZLES

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This is a study of the plane or axisymmetrical hypersonic flow of an ideal gas in an expanding nozzle whose wall approximates a parabola of degree k. The possibility of attainment of arbitrarily large Mach numbers in this flow is studied on the basis of the general equations of hypersonic flows derived in [1 and 2]. The necessary conditions to be fulfilled by k in order for isentropic expansion up to $M = \infty$ to be attainable in the nozzle are determined. If these conditions are violated, isentropic flow breaks down. Examples of self-similar solutions to illustrate the possible cases of flow are constructed with the aid of results obtained in [3 and 4].

1. Let us consider the plane or axisymmetrical isentropic hypersonic flow of an ideal perfect gas in an expanding nozzle the equation of whose surface may be written as

$$y = cx^k (1 + \Delta (x)), \qquad \lim_{x\to\infty} \Delta (x) = 0 \tag{1.1}$$

where c and k are positive constants, the Cartesian (cylindrical) coordinates x, y are expressed in fractions of some characteristic length, and the x-axis in the axisymmetrical flow is the axis of symmetry. Starting at some cross section, the flow may be considered hypersonic, since the M number increases without limit with increasing x. In investigating the flow, we make use of the general equations of hypersonic flows [1 and 2] which in the case of plane or axisymmetrical isentropic flow are of the form

$$\frac{1}{\gamma-1}\frac{\partial \ln \eta}{\partial s} + \frac{\partial \theta}{\partial n} + (\nu - 1)\frac{\sin\theta}{y} = 0, \qquad \frac{\partial \theta}{\partial s} + \frac{\partial \eta}{\partial n} = 0$$

$$\frac{\partial}{\partial s} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial n} = -\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y} \tag{1.2}$$

where γ is the adiabatic exponent, θ is the slope angle between the velocity vector and the x-axis, $\nu = 1, 2$ for plane and axisymmetrical flows, respectively, and η is a small quantity related to the local Mach number M by Expression

$$M = \sqrt{2}\,(1-\eta)\,[(\gamma-1)\,\eta\,(2-\eta)]^{-1/2} \tag{1.3}$$

As was shown in [1 and 2], the properties of arbitrary hypersonic flows depend on the value of the single parameter $K = M_* \vartheta$, where M and ϑ are the Mach number and slope angle of the velocity vector, respectively.

2. Let us trace the variation of parameter K in the flow in question. For $k \geqslant 1$ the parameter K clearly goes to infinity, since $\vartheta \sim 1$ and $M \to \infty$ as x increases. In this case, the conclusion drawn in [1 and 2] concerning the appearance of infinite domains of definition of the solution is valid. The equations given in [1 and 2] have a simple asymptotic solution of the source type; the source intensity changes in going from one flow line to another. For $k > 1$, vacuum pockets may appear near the walls. It can be shown easily that the boundary l of flow with a vacuum will be a straight line (Fig.1).

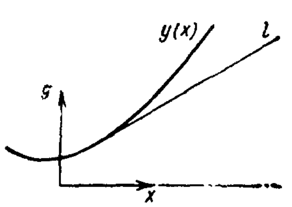

Fig. 1

Let us see how the parameter K changes for $k < 1$. From the equation of continuity written in integral form we have

$$\int_0^{y(x)} \rho u y^{\nu-1} \, dy = \text{const} \tag{2.1}$$

where u and ρ are the axial velocity and density, respectively; the integral is taken over the nozzle cross section for a fixed x. As x tends to infinity, u tends to the maximum rate of gas escape into the vacuum U_*. Assuming that no vacuum pockets form in the channel, we use Equation (2.1) to obtain the law of variation of the average density ρ_* over the cross section. Then, knowing ρ_*, we apply the isentropicity equations to find the law of variation of the remaining flow parameters – the pressure p_*, the velocity of sound a_*, and the Mach number M_* corresponding to the given value of ρ_*

$$\rho_* \sim x^{-\nu k}, \qquad p_* \sim x^{-\nu k \gamma}, \qquad a_* \sim x^{-1/2\nu k(\gamma-1)}, \qquad M_* \sim x^{1/2\nu k(\gamma-1)} \tag{2.2}$$

As the characteristic angle ϑ of the given cross section we can take the slope angle between the channel wall and the x-axis at that cross section

$$\vartheta = \tan^{-1} (dy \,/\, dx) \sim x^{k-1} \tag{2.3}$$

Hence, we have

$$K = M_* \vartheta \sim x^m, \qquad m = \frac{k}{n} - 1, \qquad n = \frac{1}{1 + 1/2\,\nu(\gamma - 1)} \tag{2.4}$$

for a parameter K which determines the flow.

The following three cases are possible:

$$1 > k > n, \qquad k = n, \qquad n > k > 0 \tag{2.5}$$

3. For $1 > k > n$, the parameter K defined by Expression (2.4) increases without limit. It is easy to show that if isentropic flow is assumed, the product $M\theta$ calculated form values at the wall also increases without limit with increasing x. Assuming the opposite, i.e. that the quantity $M\theta$ at the wall is bounded everywhere and that the flow is isentropic, we can solve the Cauchy problem and find the flow everywhere in the channel an the basis of the given values of the flow parameters at the wall.

As in the case of flow around a slender body, $M\theta$ is a bounded quantity of order of magnitude one. The quantity $K=M_*\vartheta$, turns out to be bounded in exactly the same way, which contradicts our premise.

Let us now show that shockless flow in the nozzle is impossible in the case being considered. Once again we assume the opposite: let isentropic flow take place in the nozzle up to $x = \infty$. We choose a cross section AB (Fig.2) such that the parameter $M\theta$ is sufficiently large everywhere. Let us extend the second-family characteristic AC from the point A. According to [1 and 2], this characteristic goes to infinity without intersecting the x-axis forming a constant angle θ' with the x-axis at infinity. The angle θ' is given by the expression (Equation (6.1) of [2]).

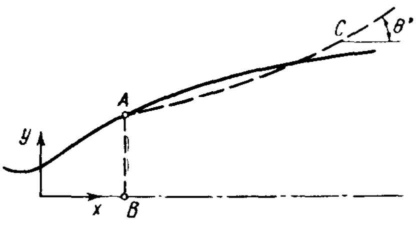

Fig. 2

$$\theta' = \vartheta_a - \frac{2}{(\gamma-1)\nu M_a} \tag{3.1}$$

where ϑ_a and M_a are the slope angle between the wall and the x-axis and the Mach number at the point A, respectively.

The quantity θ' is indeed greater than zero, since by virtue of the unlimited increase of K it is always possible to find an x such that the second term in the right side of (3.1) is less than the first. The characteristic AC therefore inevitably intersects the channel wall, and since the angle between it and the x-axis tends to zero, shock waves arise in the stream. Having assumed the flow to be isentropic, we have arrived at a contradiction, which proves that shockless flow in this case is impossible.

4. For $k = n$, the parameter K as $x \to \infty$ remains a finite quantity of order of magnitude one. Recalling that the angle of slope ϑ of the nozzle wall tends to zero with increasing x, we can take

$$\cos\theta \simeq 1, \qquad \sin\theta \approx \theta, \qquad \frac{\partial}{\partial s} = \frac{\partial}{\partial x} + \theta\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial n} = \frac{\partial}{\partial y} \tag{4.1}$$

in Equation (1.2) with a relative error of ϑ^2, whereupon Equations (1.2) are transformed [1 and 2] into a system of equations of unsteady isentropic flow in accordance with the theory of small perturbations of hypersonic flows [5 and 6],

$$\frac{1}{\gamma-1}\left(\frac{\partial\ln\eta}{\partial x} + \theta\frac{\partial\ln\eta}{\partial y}\right) + \frac{\partial\theta}{\partial y} + (\nu-1)\frac{\theta}{y} = 0, \qquad \frac{\partial\theta}{\partial x} + \theta\frac{\partial\theta}{\partial y} + \frac{\partial\eta}{\partial y} = 0 \tag{4.2}$$

Equations (4.2) coincide with the equations of one-dimensional unsteady flow if we interpret η, θ, x, y as the elasticity, velocity, time, and coordinate, respectively If the nozzle wall is represented by Equation $y = \alpha x^n$ ($\Delta(x)$ in Equation (1.1) is set identically equal to zero), then Equations (4.2) admit of a simple self-similar solution belonging to the class of solutions of the problem of fixed gas mass scattering obtained by Sedov [3]. This class of solutions was used by Nikol'skii [4] for constructing isentropic flows in nozzles and exit cones. In the special case of self-similar isentropic motion, the solution has the simple form

$$\theta = n\frac{y}{x}, \qquad \eta = \frac{0.5n(1-n)\lambda^2 + \alpha}{x^{(\gamma-1)\nu n}}, \qquad \lambda = \frac{y}{x^n} \tag{4.3}$$

where α is an arbitrary positive constant, and n is given by Equation (2.4). It is easy to see that the stagnation condition $\theta = dy/dx$ is fulfilled at the wall $y = \alpha x^n$. As was noted in [4], the distribution of flow parameters over the nozzle cross section remains nonuniform all the way to infinity, i.e. the case at hand does indeed fit within the framework of the conventional theory of small perturbations of hypersonic flow.

5. For $n > k > 0$, the parameter K tends to zero with increasing x. As was shown in [1], the ratio of the second term in the second equation of (1.2) to the first term is equal in order of magnitude to K^{-2}, i.e. as $K \to 0$ we arrive at the equation $\partial\eta/\partial n = 0$, or $\partial\eta/\partial y = 0$ for the case of elongated nozzles. This means that in the present instance it is justifiable to apply to the nozzle the hydraulic approximation wherein the enthalpy, pressure and density are uniform over the cross section. Taking into account flow rate equation (2.1) and the isentropic character of the flow, we obtain the asymptotic expressions

$$\rho = \rho_* \left(\frac{x_*}{x}\right)^{\nu k}, \qquad p = p_* \left(\frac{x_*}{x}\right)^{\nu k \gamma}, \qquad \eta = \eta_* \left(\frac{x_*}{x}\right)^{\gamma k (\gamma - 1)} \tag{5.1}$$

for p, ρ and η, where the quantities with subscript $*$ denote some fixed values.

6. By way of example, let us consider the self-similar solutions [3 and 7] of isentropic flow equations (4.2). We attempt to find a solution of the form

$$\theta = \frac{y}{x} V(\lambda), \qquad \eta = \frac{y^2}{(\gamma - 1) x^2} z(\lambda), \qquad \lambda = \frac{y}{x^\mu} \tag{6.1}$$

We obtain the system of equations

$$\frac{dz}{dV} = \frac{2z\,[z - 0{,}5\,(\gamma - 1)\,V\,(1 - V) + (V - \mu)\,(1 - V/n)]}{V\,(1 - V)\,(V - \mu) - \nu z\,(l - V)}, \qquad l = \frac{2\,(1 - \mu)}{\nu\,(\gamma - 1)} \tag{6.2}$$

$$\frac{d \ln \lambda}{dV} = \frac{(V - \mu)^2 - z}{V\,(1 - V)\,(V - \mu) - \nu z\,(l - V)} \tag{6.3}$$

Let us consider in detail the case $0 < l < \mu$ (the meaning of the parameter l will become clear below). The condition $l < \mu$, it is easy to show, implies that $l < n < \mu$. Fig. 3 shows the pattern of integral curves of Equation (6.2) for $\nu = 2$ in the half-plane $z > 0$. The arrows point in the dirction of increasing λ. The points O $(z = 0,\ V = 0)$; B $(z = 0.5(\gamma - 1)n(1 - n),\ V = n)$; D $(z = 0,\ V = 1)$ are nodes, the points A $(z = 0,\ V = \mu)$ and C $(z = \infty,\ V = l)$ are saddle points. For $\nu = 1$ we have the additional saddle point

$$E\ (z = \mu^2\,(\gamma - 1)^2\ (3 - \gamma)^{-2}$$
$$V = 2\mu\ (3 - \gamma)^{-1})$$

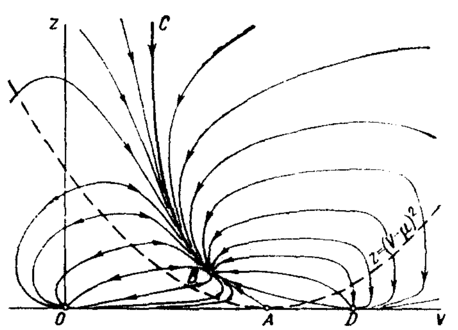

Fig. 3

For $\nu = 2$ and $1 > \mu > \mu_{*1}$, where

$$\mu_{*1} = \frac{\gamma^2 - 3\gamma + 4 + \sqrt{2}\,(\gamma - 1)^{3/2}}{2\,(\gamma^2 - 2\gamma + 2)} \tag{6.4}$$

there arise two other singular points that coincide with the points G_1 and G_2 where the curves of slope zero $(z = z_0(V))$ and infinity $(z = z_\infty(V))$ of Equation (6.2) intersect

$$\begin{aligned} z = z_0(V) &\equiv \frac{\gamma - 1}{2} V\,(1 - V) - (V - \mu)\left(1 - \frac{V}{n}\right) \\ z = z_\infty(V) &\equiv \frac{V\,(1 - V)\,(V - \mu)}{\nu\,(l - V)} \end{aligned} \tag{6.5}$$

The coordinates of the points of intersection are

$$V_{1,2} = (\gamma - 1)^{-1} \{{}^1/_2 (3 - \gamma) + \mu (\gamma - 2) \pm$$

$$\pm \sqrt{{}^1/_4 (\gamma - 3)^2 - \mu [{}^7/_4 + (\gamma - {}^3/_2)^2] + \mu^2 [(\gamma - 1)^2 + 1]}\} \tag{6.6}$$

$$z_{1,2} = (V_{1,2} - \mu)^2$$

where the subscripts 1 and 2 refer to the points G_1 and G_2, respectively.

For $\mu = \mu_{*1}$, the curves $z = z_0(V)$ and $z = z_\infty(V)$ have a point of tangency with the coordinates

$$V_1 = V_2 = \frac{1 + (\gamma - 2)\, 2^{-1/2} (\gamma - 1)^{1/2}}{\gamma^2 - 2\gamma + 2}, \qquad z = (V_{1,2} - \mu_{*1})^2 \tag{6.7}$$

We note that the point of tangency appears to the left of the point B for $\gamma < 2$ and to the right of point B for $\gamma > 2$ (Fig.3 corresponds to the case $\mu < \mu_*$ when the points $V_{1,2}$, $z_{1,2}$ are lacking). For $\mu < \mu_{*1}$, the point B clearly lies above the curve $z = (V - \mu)^2$, while for $\mu > \mu_{*1}$ it may turn out to lie below this curve. The curve $z = (V - \mu)^2$ corresponds to a limiting line in the physical plane, since the direction of change of λ is reversed in passing over it [3 and 8]. The limiting line, in addition to the point A, where it intersects the curve $z = z_\infty(V)$, for $\nu = 1$ also passes through the point E ($z = \mu^2(\gamma - 1)^2(3 - \gamma)^{-2}$, $V = 2\mu(3 - \gamma)^{-1}$), and for $\nu = 2$ – through the points G_1, G_2 (Equation (6.6)) if these are present. The required integral curve passes through the saddle point C, in whose neighborhood it can be represented as

$$V = \sum_{i=0}^{\infty} \frac{a_i}{z^i}, \qquad a_0 = l, \qquad a_1 = \frac{l(1 - l)(\mu - l)}{2 + \nu}, \ldots \tag{6.8}$$

From Equation (6.3), taking into account (6.8), we find that

$$\lambda = \frac{q}{\sqrt{z}} \left[1 - \frac{(l - \mu)^2 + bl - \mu - al^2}{2z} \ \ldots \right] \tag{6.9}$$

$$a = 1 + {}^1/_2 (\nu - 1)(\gamma - 1), \qquad b = 1 + \mu + {}^1/_2 (\mu\nu - 1)(\gamma - 1)$$

where q is an integration constant. Making use of Equations (6.1), (6.8) and (6.9), we obtain

$$\theta = l \frac{y}{x} (1 + \omega_2 \lambda^2 + \omega_4 \lambda^4 + \ldots), \qquad \eta^2 = \frac{q^2}{(\gamma - 1) x^{2(1-\mu)}} (1 + \delta_2 \lambda^2 + \delta_4 \lambda^4 + \ldots) \tag{6.10}$$

where the constants ω_i and δ_i are expressed in terms of the constant q and the coefficients of expansion of Equations (6.8) and (6.9). For $\nu = 1$ (or for $\nu = 2$ and $\mu < \mu_{*1}$), a solution of (6.8), (6.9) and (6.10) constructed in the neighborhood of the point C can be extended to the singular point B that lies above the curve $z = (V - \mu)^2$, as shown in Fig.3. In physical coordinates the value $\lambda = \infty$ corresponds to this singular point. Let us take some point A in the xy-plane (Fig.4) and pass through it the streamline whose equation is

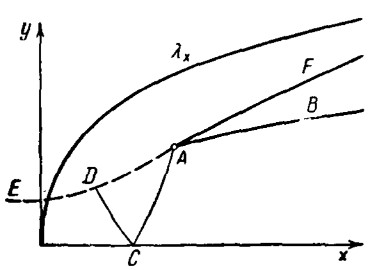

Fig. 4

$$\frac{dy}{dx} = \frac{y}{x} V \quad (l \leqslant V \leqslant n) \tag{6.11}$$

The limits of variation of V are apparent from Fig.3. Equation (6.11) allows us to establish the inequality

$$y(x) < Nx^n \tag{6.12}$$

for the streamline $y = y(x)$, where n is some constant. The quantity $\lambda = yx^{-\mu} < Nx^{n-\mu}$ therefore tends to zero with increasing x in the region of flow between the streamline AB and the x-axis (Fig.4), since we are considering the case where $\mu > n$. To find the equation of the streamline, we make use of the first equation of (6.10),

$$\frac{dy}{dx} = l\,\frac{y}{x}\,(1 + \omega_2\lambda^2 + \omega_4\lambda^4 + \ldots), \qquad y = cx^l\,[1 + \alpha_2 x^{2(l-\mu)} + \alpha_4 x^{4(l-\mu)} + \ldots] \tag{6.13}$$

where c is an integration constant, and α_i are constants that depend on ω_i and c. Since by hypothesis $l < \mu$, the second equation of (6.13) can be written in the form (1.1), where we set $l = k$, and

$$\Delta(x) = \alpha_2 x^{2(k-\mu)} + \alpha_4 x^{4(k-\mu)} + \ldots, \quad \lim_{x\to\infty}\Delta(x) = 0 \tag{6.14}$$

Extending the characteristic of the first family AC (Fig.4) through the point A and solving the Goursat problem in the characteristic triangle ACD, the solution thus constructed can be "sewn" to any solution for isentropic flow in channel DE. CD is the characteristic of the second family for flow in the channel. The second equation of (6.10) can be written as

$$\eta = \frac{q^2}{(\gamma - 1)\,x^{\nu k(\gamma-1)}}\,[1 + \delta(x, y)], \qquad \lim_{x\to\infty}\delta(x, y) = 0 \tag{6.15}$$

by taking into account Expression (6.2) for l.

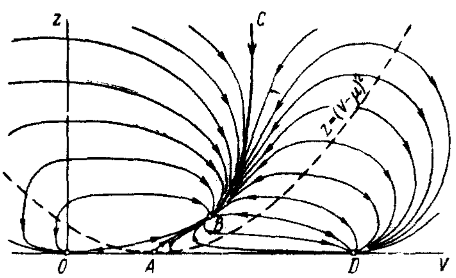

Fig. 5

Comparing the last equation of (5.1) with (6.15), we find confirmation of Section 5 in that the flow under consideration obeys the laws of hydraulics.

If the point B lies below the curve $z = (V - \mu)^2$ (this is possible for $\nu = 2$ and $\mu > \mu_{*1}$), an integral curve extended from the point C passes through the node G_2 with coordinates (6.6) and may then intersect the curve $z = (V - \mu)^2$. In this case a limiting line $\lambda = \lambda_*$ appears in the flow region (Fig.4). It is clear that if the point A is chosen in the region $\lambda < \lambda_*$, a streamline extended through the point A does not intersect the limiting line with increasing x.

7. For $l = \mu = n$ Equations (6.2) and (6.3) have the simple solution $V = n$, $z = 0.5(\gamma - 1)n(1 - n)$, which is a special case of solution (4.3).

8. The pattern of integral curves for $1 > l > n > \mu$ and $\nu = 1$ (or for $\nu = 2$ and $\mu > \mu_{*2}$) is shown in Fig.5.

Here

$$\mu_{*2} = \frac{\gamma^2 - 3\gamma + 4 - \sqrt{2}\,(\gamma - 1)^{3/2}}{2\,(\gamma^2 - 2\gamma + 2)} \tag{8.1}$$

As before, the required integral curve connects the points C ($z = \infty$, $V = l$) and B ($z = 0.5(\gamma - 1)n(1 - n)$, $V = n$). The point B corresponds to $\lambda = \infty$ in the physical plane. It is easy to show that the parameter λ increases without limit with increasing x along the streamline extanded from any point A in the xy-plane. For this reason, the asymptotic representation of the streamline equation obtained by integration (6.11), recalling that $V \to n$ as $\lambda \to \infty$, may be written in the form

$$y = cx^n\,(1 + \Delta(x)), \qquad \lim_{x\to\infty}\Delta(x) = 0 \tag{8.2}$$

where c is some constant. The flow we are considering, as in Section 7 belongs to the class of flows considered in Section 4. The nonuniform distribution of flow parameters over the nozzle cross section continues all the way out to infinity without any disruption of potential flow.

If the point B lies below the parabola $z = (V-\mu)^2$, as may happen for $\nu = 2$ and $\mu < \mu_{*2}$, an integral curve extended from the point C passes through the node G_1 with coordinates (6.6). It can be shown that the point G_1 corresponds to a characteristic of the second family in the physical plane, so that weak discontinuities of the flow parameters may occur on it. Hence, an integral curve entering the node G_1 from above can be extended by any integral curve emerging from that node. As shown in Fig.6, integral curves of three types are possible: z_1, z_2 and z_3 (z_0 and z_∞ are given by Equation (6.5)). The curve z_2 passes through the point B, which in this case happens to be a saddle point. The asymptotic representation of the streamline is of the form (8.2) as before. The curve z_3 entering the node D ($z = 0, V = 1$) describes a flow that turns into inertial scattering of the gas for $\lambda = \infty$. The curve z_1 corresponds to flow in which any streamline (AF in Fig.3) with increasing x inevitably intersects the limiting line with resulting disruption of flow. These examples confirm the conclusion drawn in Section 3 as regards the impossibility of shockless flow in a nozzle with generator (1.1) for $1 > k > n$.

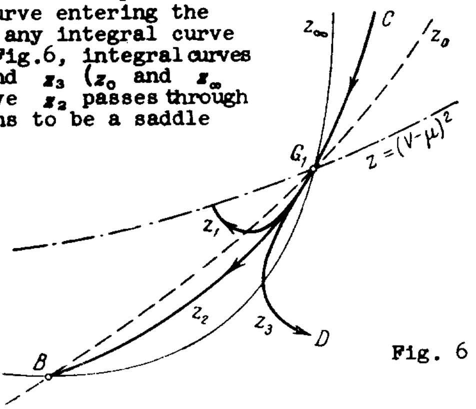

Fig. 6

9. For $l = 1$, Equations (6.2) and (6.3) have the simple solution $V = 1$, $z = 0$ corresponding to inertial scattering of the gas particles. For $l > 1$, the streamlines inevitably intersect the limiting line and the flow is disrupted. For this reason, the stream cannot be accelerated up to $M = \infty$ more rapidly than in the case of a hypersonic source; this is in accord with the results of Section 2. The illustrations given here therefore confirm the conclusions of Sections 2 to 5 about the construction of isentropic flow in an expanding nozzle.

We note that for large numbers M, a thick boundary layer usually arises in nozzles; this layer tends to squeeze out flow in the isentropic core of the stream. In the present paper we have indicated what the law of expansion of the potential core must be in order for shock waves not to arise in the nozzle.

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